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LETTER TO THE EDITOR

Discussion of the article "Vibration of pretwisted cantilever shallow conical shells", *Int. J. Solids Structures* Vol. 31, No. 18, pp. 2463–2476 (1994), by K. M. Liew, C. W. Lim and L. S. Ong

This article alleges to calculate the vibrational frequencies of various twisted turbine blades by a Rayleigh–Ritz method with polynomial expansions. However, the article contains errors in principle with regard to the use of shallow shell theory which render questionable almost all of the results. This letter is intended to discuss these errors. We begin with a brief discussion of general shell theory principles followed by a statement of the assumptions behind the shallow shell theory approximation. In this letter, we are not proposing a correction to the authors' analysis; rather, we are pointing out the violations of the shallow shell assumptions.

The most general form of shell theory starts with a coordinate system (ζ_1, ζ_2) embedded in the middle surface of the shell, which lies in a larger Euclidean space described by Cartesian coordinates (x, y, z). The formulation of displacements, membrane strains, curvatures, stress resultants, and bending moments, and the derivation of the governing equations is done entirely with respect to the intrinsic middle surface coordinates. For example, the displacement vector, as measured from a global system of Cartesian coordinates, is resolved into components pointing in the local middle surface coordinate system. Other quantities are treated in a similar manner. The final boundary value problem is given in terms of (ζ_1, ζ_2).

In certain cases where the radii of curvature of the undeformed shell are much larger than the characteristic lengths in the middle surface, the shell is plate-like, and shallow shell theory may be invoked to transform the problem from the middle surface coordinates into coordinates lying in the basal plane onto which the undeformed shell is projected. This process imposes several restrictions which the twisted, tapered blade geometries of this study do not satisfy.

In the shallow shell theory approximation, the reference configuration of the middle surface is written in the form

$$\mathbf{R}(x, y) = \mathbf{r}(x, y) + z(x, y)\mathbf{k} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

where the coordinates (x, y) are a set of basal plane coordinates (as in fact adopted by the authors). The metric tensor in these coordinates is given by

$$g_{\alpha\beta} = (g_0)_{\alpha\beta} + z_{,\alpha} z_{,\beta}$$

where $(g_0)_{\alpha\beta}$ is the metric in the basal plane. The curvature tensor is expanded in a similar manner. When complete, the assumptions behind the theory are:

(1) The metric tensor $g_{\alpha\beta} \sim (g_0)_{\alpha\beta}$, which is subject to the shallowness restriction $(g_0)^{\alpha\beta} z_{,\alpha} z_{,\beta} \ll 1$. This is a fundamental requirement.
(2) The curvature tensor $b_{\alpha\beta}$ of the undeformed middle surface is given by $-z_{,\alpha\beta}$.
(3) Covariant differentiation in the middle surface plane $()_{;}$ is equivalent to covariant differentiation in the basal plane $()_{,}$. This assumption is closely linked with the first assumption since the Christoffal symbols are derived from the derivatives of the metric tensor.
(4) The strain tensor is given by

$$E_{\alpha\beta} = \tfrac{1}{2}(u_{\alpha,\beta} + u_{\beta,\alpha}) - wz_{,\alpha\beta}$$

where u_α are the basal plane displacements and w is the displacement normal to the basal plane.

(5) The bending curvature tensor is the same as in plate theory,

$$K_{\alpha\beta} = -w_{,\alpha\beta}.$$

This requires that $|u_\alpha|/R \ll O(w/L)$, where L is the characteristic in-plane length of the vertical deflection and R is the *smallest* principal radius of curvature of the undeformed middle surface. In addition, this form of the linear bending curvature tensor is only valid when $L \ll R$, otherwise additional terms appear (Niordson, 1985).

The first problem is that the authors do not seem to appreciate the basic coordinate system assumptions behind shallow shell theory, which is clearly evidenced in the first full sentence on p. 2465:

> "The displacements are resolved into three orthogonal components u, v, and w with respect to the midsurface of the shell with u along the x-axis, v tangential to the midsurface, and w normal to it."

However, this statement itself is not consistent with the authors' subsequent analysis, which does appear to measure the displacements with respect to the basal plane.

The first error in this analysis is the violation of the shallowness assumption (1). This is most clearly demonstrated for *untwisted* cylindrical blades, where the opening angle θ_0 in the authors' terminology can be shown to be limited by the requirement that $\tan^2(\theta_0/2) \ll 1$. If an error $O(0.01)$ is adopted, then $\theta_0 \sim 12°$. The use of $\theta_0 = 30°$ throughout most of the authors' results gives $\tan^2(15°) \sim 0.07$, which is not really $\ll 1$. The superposition of an axial twist of ψ on top of the cylindrical panel results in the restriction that $\tan^2(\psi + \theta_0/2) \ll 1$, which is totally violated by the authors' subsequent analyses of twist angles as large as $\psi = 45°$. The shallow shell approach is good *only* for small amounts of twist on cylindrical blades with small opening angles. This argument is just as valid, although a bit more complicated, for the authors' conical blades. The authors have even cited a form of this restriction in Lim and Liew (1994); however, they have not adhered to it.

The second major error in this analysis is the violation of the conditions associated with assumption (5). For the "long" cantilevered beams studied (see the authors' Table 2), the fundamental mode is a bending-type motion, as in fact suggested by the authors' own mode shapes. However, the analysis in this paper will not accurately capture the lowest frequency since the characteristic length, $L = a$, is *greater than* R in many of the authors' geometries, and this is a clear violation of the assumptions behind (5).

The form of the bending curvature tensor in assumption (5) is associated with the often cited restriction that the stresses due to bending are *less than or equal to* the stresses due to membrane forces for shallow shell and Donnell–Mushtari–Vlasov theories to be accurate. It is inappropriate to apply these theories for cantilevered shells since the bending stresses then greatly dominate membrane stresses.

The serious effect of the authors' error on their results can be detected by suppressing all in-plane displacements and derivatives in the y-direction, and then taking the variation of their energy functional to obtain the differential equation

$$\frac{d^4w}{dx^4} + \frac{12w}{(hR)^2} = \frac{\rho\omega^2}{D}w.$$

When the characteristic length $L = a$ along with the dimensionless variables $\bar{w} = w/L$ and $\bar{x} = x/L$ are introduced, this equation becomes

$$\frac{d^4\bar{w}}{d\bar{x}^4}+12\bar{w}\left(\frac{L^2}{hR}\right)^2=\frac{\rho\omega^2L^4\bar{w}}{D}.$$

In many of the geometries considered by the authors, the dimensionless parameter in the second term of this equation has a value in excess of 1000 which greatly dominates the leading bending term and so the analysis cannot recapture the "beam-like" behavior of these geometries. This inaccuracy is a direct result of improper kinematic assumptions.

A third error can be seen in the authors' eqns (5) on p. 2465, which are not consistent with the assumption (4) on the form of the strain tensor. There appears to be several incomplete and missing terms describing the $z_{,\alpha\beta}$ tensor for the twisted conical blade. As a result, the approach may well fail even for conical blade geometries that are amenable to the shallow shell analysis.

We close with some examples of the inaccuracy of these results in comparison with those given by finite element analyses. For numerical comparison, we study several geometries corresponding to results given in Table 2 on p. 2471. The values selected are: $\theta_0 = 30°$, $\theta_v = 15°$, $s/h = 1000$. Four lengths $a/s = 0.2$, 0.3, 0.5 and 0.8 for four twists $\psi = 0°$, 15°, 30° and $\psi = 45°$ are considered. (Such a large value of the thickness parameter s/h makes one wonder what possible practical application the authors had in mind.)

Our calculations were conducted with two commercial finite element packages: LUSAS and STRAND6. The analyses employed 8-node semi-loof elements. Each element has 32 degrees of freedom: three displacements per node and two loof rotations on each side spaced between nodes. The meshes, in order of increasing a/s, contained 10×30, 8×35, 7×40, and 6×60 elements. The elements were "almost" square in shape. The mesh results have been verified against a range of vibration problems and checked for convergence. Results for "long" untwisted geometries were also compared to beam theory predictions. Table 1 shows the LUSAS calculations for the first two dimensionless frequency parameters (corresponding to those in the authors' Table 2), together with the percentage differences of the authors' values with respect to our values shown in parentheses—a positive value indicates the authors' results are higher than ours.

The demonstration of the authors' mistake is in the long ($a/s = 0.8$) *untwisted* blade results, which differ significantly from the finite element calculations. The fact that the percentage difference with the finite-element results increases with length is a clear example

Table 1. Values of frequency parameter $\lambda_i = \omega_i(b_0)^2\sqrt{(\rho h/D)}$ for the first two modes of a twisted conical shell clamped at one end, with $\theta_0 = 30°$, $\theta_v = 15°$, $s/h = 1000$. The results were obtained with the LUSAS finite element package, with values in brackets being the percentage difference of Liew & Lim's results from these

ϕ	a/s	λ_1 (% error)	λ_2 (% error)
0°	0.2	1.9047 (10.3%)	3.0518 (1.2%)
	0.3	0.86749 (18.4%)	2.1482 (1.3%)
	0.5	0.32779 (39.4%)	1.5651 (1.4%)
	0.8	0.14552 (91.3%)	0.55036 (29.4%)
15°	0.2	1.3861 (6.6%)	4.0388 (4.1%)
	0.3	0.73216 (15.3%)	2.4906 (3.4%)
	0.5	0.30865 (37.1%)	1.4654 (10.9%)
	0.8	0.14299 (88.6%)	0.53595 (30.1%)
30°	0.2	0.97675 (−3.3%)	4.3306 (3.0%)
	0.3	0.55570 (4.7%)	2.5599 (4.5%)
	0.5	0.26823 (26.7%)	1.2420 (10.0%)
	0.8	0.13624 (77.5%)	0.49817 (30.9%)
45°	0.2	0.77081 (−20.0%)	3.5265 (−7.3%)
	0.3	0.44180 (−15.1%)	2.0265 (−6.6%)
	0.5	0.22871 (3.3%)	1.0086 (1.9%)
	0.8	0.12713 (50.0%)	0.44816 (28.4%)

of the violation of the geometric restriction that $L \ll R$. The *twisted* blade results violate the shallowness assumption and are also significantly different from the finite element results.

The error in this analysis are serious enough that all of the calculations must be viewed with scepticism. There may be sufficient information in this incorrect formulation to capture some of the physical trends for cantilevered twisted and untwisted shells; however, this should be regarded as coincidental.

DAVID M. STUMP, GRAHAM BAKER
Department of Civil Engineering
The University of Queensland
St Lucia QLD 4072
Australia

GRANT P. STEVEN
Department of Aeronautical Engineering
The University of Sydney
NSW 2006
Australia

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